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Open-string actions and noncommutativity beyond the large-B limit

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Abstract: In the limit of large, constant B-field (the "Seiberg-Witten limit"), the derivative expansion for open-superstring effective actions is naturally expressed in terms of the symmetric products $*_n$. Here, we investigate corrections around the large-B limit, for Chern-Simons couplings on the brane and to quadratic order in gauge fields. We perform a boundary-state computation in the commutative theory, and compare it with the corresponding computation on the noncommutative side. These results are then used to examine the possible role of Wilson lines beyond the Seiberg-Witten limit. To quadratic order in fields, the entire tree-level action is described by a metric-dependent deformation of the $*_2$ product, which can be interpreted in terms of a deformed (non-associative) version of the Moyal $*$ product.

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Contents

1. Introduction

In the background of a constant and large B-field, the structure of higher-derivative terms in the open-superstring effective action is encoded in certain symmetric products $*_n$, as follows from refs. [1]–[8].[1] The higher-derivative terms that have been understood in this way are those which are leading at large B. This corresponds to the the Seiberg-Witten (SW) limit [9], which is effectively the same as the limit of very large B-field [10]. In the present work, we will extend this understanding to terms that are subleading at large B-field, or equivalently to corrections beyond the Seiberg-Witten limit.

This approach to derivative corrections originally grew out of the comparison between open-string actions in commutative and noncommutative variables. The $*_2$ product was discovered in ref. [1], and was generalized to $*_n$ and extensively studied in refs. [11, 2, 3]. Following this, noncommutative Ramond-Ramond couplings were studied, and comparison with commutative Ramond-Ramond couplings [4]–[6] led to new topological identities on noncommutative gauge fields, as well as a derivation of an explicit expression for the Seiberg-Witten map [9], previously conjectured in ref. [2].

[1] Here and in the following, "open-string actions" will mean the tree-level low-energy effective action for the coupling of open-string modes to linearized closed-string modes. We will always take the closed-string fields to be at arbitrary nonzero momentum. We will work with a single euclidean D9-brane with a constant B-field of maximal rank. The extension to D-branes of lower dimension is straightforward. However, the nonabelian case involving multiple D-branes seems much more difficult, and is beyond the scope of this paper.

Independently, higher derivative corrections to the commutative open-superstring effective action (including couplings to both RR and NS-NS closed string fields) were computed in ref. [12] using the boundary-state formalism. These computations led to elegant expressions that suggested an underlying mathematical structure. This structure was later revealed [7] to arise from the $*_n$ product, via the relation between noncommutative and commutative actions. Some of these computations were extended to higher orders in ref. [13].

Both commutative and noncommutative actions are infinite power series in spacetime derivatives. Since derivatives carry a spatial dimension, this has to be cancelled by a suitable dependence on the available dimensional constants, which are α' and B. However, the two actions organize these series differently. In particular, the dependence of the commutative action on B arises through the generalized inverse metric

$$h^{ij} \equiv \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij} \tag{1.1}$$

while in the noncommutative action the dependence arises through two kinds of sources. One is the noncommutativity parameter θ arising in the Moyal $*$-product:

$$f(x) * g(x) \equiv f(x)\, e^{(i/2)\overleftarrow{\partial_p}\,\theta^{pq}\,\overrightarrow{\partial_q}}\, g(x) \tag{1.2}$$

while the other is the dependence of the action on the open-string metric G_{ij} and coupling G_s.

The expressions for θ and G in terms of closed-string quantities depend on an antisymmetric matrix called the "description parameter" [9, 16], denoted Φ_{ij}, and we have:

$$\left(\frac{1}{G+2\pi\alpha'\Phi}\right)^{ij} + \frac{\theta^{ij}}{2\pi\alpha'} = \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}. \tag{1.3}$$

The natural low-energy limit $\alpha' \to 0$ for open strings is called the Seiberg-Witten (SW) limit. In this limit, G, B, Φ are kept fixed. It follows from the above equation that $g_{ij} \sim \alpha'^2$ and so the closed-string metric goes to zero faster than $\alpha' B$. In this sense it is a large-B limit.

Expanding about this limit, we find that:

$$\begin{aligned}
\theta^{ij} &= \left(\frac{1}{B}\right)^{ij} + \mathcal{O}(\alpha') \\
G^{ij} &= -\frac{\theta^{ik} g_{kl} \theta^{lj}}{(2\pi\alpha')^2} + \mathcal{O}(\alpha')
\end{aligned} \tag{1.4}$$

Note that the leading terms do not depend on the description parameter Φ, only the subleading corrections depend on it. For a particular value $\Phi = -B$, the corrections in the above equation vanish to all orders.

To compute derivative corrections, to high orders in derivatives, has traditionally been a difficult technical exercise.[2] It turns out that noncommutativity is a handy tool to obtain

[2]Earlier work on derivative corrections in open-string theory can be found in ref. [14].

large parts of the result. In the limit of large B, one makes the replacement:

$$h^{ij} \to \frac{\theta^{ij}}{2\pi\alpha'} \,. \tag{1.5}$$

On the commutative side this gives rise to infinitely many derivative corrections that, after the above replacement, are all of lowest order in α' — though they originally arose from terms of arbitrary order in α'. On the noncommutative side, this limit suppresses all but the leading zero-derivative term (DBI or Chern-Simons), and the dependence on θ is encoded in the Moyal product. The closed-string modes in the action are taken to have arbitrary nonzero momentum (otherwise the terms we are computing would all vanish upon partial integration). Hence one has to make use of the open Wilson line [15] prescription to get a gauge invariant result, leading to additional dependence on θ. The result, described in detail in ref. [7], is that the derivative expansion at large B is encoded in the $*_n$ products.[3]

As a check, predictions from noncommutativity were compared to the existing perturbative open-string amplitudes in the literature. Striking agreement was found between these predictions and the coefficients and tensor structures computed explicitly in ref. [12] to low orders in derivatives.

Inspired by this agreement and the elegant predictions of noncommutativity, a new perturbative amplitude calculation was performed in ref. [8]. Here noncommutativity was not invoked, but a specific class of derivative corrections was computed in the commutative framework, to leading order in large B and all orders in derivatives. This was the correction to the Chern-Simons (CS) coupling

$$S^{(6)}_{CS} = \frac{1}{2}\int C^{(6)} \wedge F \wedge F \,, \tag{1.6}$$

where $C^{(6)}$ is the Ramond-Ramond 6-form potential. (Recently, this calculation was extended to other couplings on the D-brane worldvolume [17].)

In the Seiberg-Witten limit ($\alpha' \to 0$ with $g \sim \alpha'^2$) it was shown by an explicit boundary-state computation that the above coupling is corrected to:

$$S^{(6)}_{CS} = \frac{1}{2}\int C^{(6)} \wedge \langle F \wedge F \rangle_{*_2} \,, \tag{1.7}$$

where

$$\langle F_{ij}(x), F_{kl}(x)\rangle_{*_2} \equiv F_{ij}(x) \frac{\sin((1/2)\overleftarrow{\partial_p}\,\theta^{pq}\,\overrightarrow{\partial_q})}{(1/2)\overleftarrow{\partial_p}\,\theta^{pq}\,\overrightarrow{\partial_q}} F_{kl}(x) \,. \tag{1.8}$$

This result holds to quadratic order in F, with higher-order corrections. The above expression agrees perfectly with a prediction from noncommutativity that was made in ref. [7]. This prediction and the explicit computation which it confirms are reviewed in section 2, to establish the techniques and notation for the subsequent sections.

One striking feature of eq. (1.7) is that it describes a set of derivative corrections all of which have rational numerical coefficients, which come from expanding the definition of $*_2$ above. Indeed, going beyond the specific CS coupling above, it follows from ref. [7] that

[3] Previous attempts to constrain effective actions using noncommutativity can be found in ref. [26].

the entire leading behaviour of open-string theory at large B-field is given by terms with rational coefficients that come from the expansion of $*_n$ products along with determinants, denominators and Wilson lines. This means that the transcendental coefficients of string theory, such as $\zeta(2n+1)$, must all drop out in this limit.[4] It would be interesting to find an independent reason for this.

Thus it is tempting to ask what happens if we go beyond the Seiberg-Witten approximation. For example, in the commutative theory, to first order in α' one would make the replacement:

$$h^{ij} \equiv \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij} \sim \frac{\theta^{ij}}{2\pi\alpha'} - \frac{(\theta g \theta)^{ij}}{(2\pi\alpha')^2} \,. \tag{1.9}$$

It is important to keep in mind that the second term is of order α' relative to the first one, because of the scaling of $g \sim \alpha'^2$ implicit in the Seiberg-Witten limit. We can also write:

$$h^{ij} \equiv \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij} \sim \frac{\theta^{ij}}{2\pi\alpha'} + G^{ij} \,. \tag{1.10}$$

From eq. (1.3) we see that this equation is exact in the $\Phi = 0$ description.

In first subleading order, it is reasonable to expect that transcendental coefficients reappear in the computation of amplitudes. This would mean that we are no longer working with an excessively over-simplified limit of open-string theory. It turns out that the perturbative calculation of ref. [8] which led to the result in eq. (1.7) can be extended without too much trouble to order α', and we will describe this computation below.

An exact tree-level computation of the three-point function of two gauge fields with a Ramond-Ramond potential in noncommutative string theory was performed in ref. [18]. This can be expanded to order α' and compared with the above result. The two computations will be seen to agree perfectly.

The paper is organized as follows. In section 2, we review the boundary-state computation of open-string actions, and the computation of the CS coupling to leading order in the large-B limit performed in ref. [8]. All the computations discussed here and subsequently are carried out in an approximation where we restrict to quadratic order in gauge potentials, but all orders in derivatives. In section 3 we use the boundary-state formalism to compute the first correction around the large-B limit. In section 4 this result is compared with a computation of Liu and Michelson [18] of the corresponding amplitude on the noncommutative side. In section 5, we use the computation of ref. [18] to demonstrate that the entire tree-level amplitude involving two open-string states and one RR potential is encoded in a deformed $*_2$ product. This in turn can be seen to arise from a deformed (non-associative) version of the Moyal $*$ product, and a standard prescription involving straight open Wilson lines. Finally, in an appendix we give direct world-sheet derivations of the proposals made in section 5.

2. Predictions from noncommutativity and boundary-state computations

This section is a review of refs. [7, 12, 8]. Let us focus on a particular Chern-Simons

[4] We thank Ashoke Sen for stressing this point.

coupling, the one involving the Ramond-Ramond 6-form $C^{(6)}$. In the commutative theory, this is just

$$\frac{1}{2}\int C^{(6)} \wedge F \wedge F\,. \tag{2.1}$$

As shown in ref. [25], the noncommutative version of this coupling is obtained from the commutative one by making the replacement:

$$F \to \hat{F}\frac{1}{1-\theta\hat{F}}\,, \tag{2.2}$$

where $\hat{F}$ is the noncommutative gauge field strength, multiplying the action by a factor $\sqrt{\det(1-\theta\hat{F})}$, and using the Moyal $*$-product defined in eq. (1.2).

Finally, to make a coupling that is gauge-invariant even for nonconstant fields, this has to be combined with an open Wilson line [2, 3]. The resulting expression [4, 5, 6, 19] for the coupling to $C^{(6)}$ is more conveniently expressed in momentum space, where $\widetilde{C}^{(6)}(k)$ is the Fourier transform of $C^{(6)}(x)$:

$$\frac{1}{2}\widetilde{C}^{(6)}(-k) \wedge \int L_* \left[\sqrt{\det(1-\theta\hat{F})}\left(\hat{F}\frac{1}{1-\theta\hat{F}}\right) \wedge \left(\hat{F}\frac{1}{1-\theta\hat{F}}\right) W(x,C)\right] * e^{ik.x}\,. \tag{2.3}$$

Here $W(x,C)$ is an open Wilson line, and L_* is the prescription of smearing local operators along the Wilson line and path-ordering with respect to the Moyal product. Evaluation of the L_* prescription leads to $*_n$ products [2, 3]. While expressions such as the above were originally obtained in the $\Phi = -B$ description, it has been argued (see refs. [3, 19, 6]) that they actually hold in all descriptions, provided the value of θ appropriate to the given description is used in the $*$ products.

In the approximation where we retain only terms that are quadratic in F, one can ignore the Wilson line, the denominators, the pfaffian prefactor and the Seiberg-Witten map relating $\hat{F}$ to F. Then the above expression can easily be re-expressed in position space, and it turns into:

$$\frac{1}{2}\int C^{(6)} \wedge \langle F \wedge F\rangle_{*_2}\,. \tag{2.4}$$

Let us now see how this result can be obtained directly by a computation in commutative string theory. The computation will be done in the boundary-state formalism. Some background on how to compute derivative corrections in this formalism may be found in ref. [12], and we will largely follow the notation in that paper.

Let us denote the sum of all derivative corrections to S_{CS} as ΔS_{CS}. Our starting point is the expression

$$S_{CS} + \Delta S_{CS} = \langle C|e^{-\frac{i}{2\pi\alpha'}\int d\sigma d\theta D\phi^i A_i(\phi)}|B\rangle_R\,, \tag{2.5}$$

where $|C\rangle$ represents the RR field, and $|B\rangle_R$ is the Ramond-sector boundary state for zero field strength. We are using superspace notation, for example $\phi^i = X^i + \theta\psi^i$ and D is the supercovariant derivative.

Combining [12, eqs. (2.3), (2.6) and (2.13)], we can rewrite this as:

$$\begin{aligned} S_{CS} + \Delta S_{CS} = {} & \langle C|e^{\frac{i}{2\pi\alpha'}\int d\sigma d\theta \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\frac{k+1}{k+2}D\widetilde{\phi}^j\widetilde{\phi}^i\widetilde{\phi}^{a_1}\ldots\widetilde{\phi}^{a_k}\partial_{a_1}\ldots\partial_{a_k}F_{ij}(x)} \times \\ & \times e^{\frac{i}{2\pi\alpha'}\int d\sigma[\widetilde{\Psi}^i\psi_0^j+\psi_0^i\psi_0^j]\sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{X}^{a_1}\ldots\widetilde{X}^{a_k}\partial_{a_1}\ldots\partial_{a_k}F_{ij}(x)}|B\rangle_R\,, \end{aligned} \tag{2.6}$$

where nonzero modes have a tilde on them, while the zero modes are explicitly indicated.

Since we are looking for couplings to the RR 6-form $C^{(6)}$, and working to order F^2, we only need terms with the structure $\partial \ldots \partial F \wedge \partial \ldots \partial F$. For such terms, two F's and 4 ψ_0's must be retained. Thus we can drop the first exponential factor in eq. (2.6) above, as well as the first fermion bilinear $\widetilde{\Psi}^i \psi_0^j$ in the second exponential. Then, expanding the exponential to second order, we get:

$$\begin{aligned} S_{CS} + \Delta S_{CS} = & \frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{i}{2\pi\alpha'} \right)^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 \, \times \\ & \times \Bigg\langle C| \left(\frac{1}{2} \psi_0^i \psi_0^j \right) \left(\frac{1}{2} \psi_0^k \psi_0^l \right) \frac{1}{n!} \widetilde{X}^{a_1}(\sigma_1) \cdots \widetilde{X}^{a_n}(\sigma_1) \frac{1}{p!} \widetilde{X}^{b_1}(\sigma_2) \cdots \widetilde{X}^{b_p}(\sigma_2) \times \\ & \times \partial_{a_1} \ldots \partial_{a_n} F_{ij}(x) \, \partial_{b_1} \ldots \partial_{b_p} F_{kl}(x) |B \Bigg\rangle_R . \end{aligned} \tag{2.7}$$

Now we need to evaluate the 2-point functions of the $\widetilde{X}$. The relevant contributions have non-logarithmic finite parts [12] and come from propagators for which there is no self-contraction. This requires that $n = p$. Then we get a combinatorial factor of $n!$ from the number of such contractions in $\langle (\widetilde{X}(\sigma_1))^n \, (\widetilde{X}(\sigma_2))^n \rangle$. The result is:

$$\begin{aligned} S_{CS} + \Delta S_{CS} = & \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2\pi\alpha'} \right)^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 \; D^{a_1 b_1}(\sigma_1 - \sigma_2) \cdots D^{a_n b_n}(\sigma_1 - \sigma_2) \; \times \\ & \times \partial_{a_1} \ldots \partial_{a_n} F_{ij}(x) \, \partial_{b_1} \ldots \partial_{b_n} F_{kl}(x) \left\langle C| \left(\frac{1}{2} \psi_0^i \psi_0^j \right) \left(\frac{1}{2} \psi_0^k \psi_0^l \right) |B \right\rangle_R . \end{aligned} \tag{2.8}$$

The fermion zero mode expectation values are evaluated using the recipe:

$$\frac{1}{2} \psi_0^i \psi_0^j F_{ij} \rightarrow (-i\alpha') F \,, \tag{2.9}$$

where the F on the right hand side is a differential 2-form. The justification for this can be found below eq. (B.3) of ref. [12]. Thus we are led to:

$$S_{CS} + \Delta S_{CS} = T^{a_1 \ldots a_n ; b_1 \ldots b_n} \, \partial_{a_1} \ldots \partial_{a_n} F \wedge \partial_{b_1} \ldots \partial_{b_n} F \,, \tag{2.10}$$

where

$$T^{a_1 \ldots a_n ; b_1 \ldots b_n} \equiv \frac{1}{2} \frac{1}{n!} \left(\frac{i}{2\pi\alpha'} \right)^2 (-i\alpha')^2 \int_0^{2\pi} d\sigma_1 \int_0^{2\pi} d\sigma_2 \, D^{a_1 b_1}(\sigma_1 - \sigma_2) \cdots D^{a_n b_n}(\sigma_1 - \sigma_2) \,. \tag{2.11}$$

Next we insert the expression for the propagator:

$$D^{ab}(\sigma_1 - \sigma_2) = \alpha' \sum_{m=1}^{\infty} \frac{e^{-\epsilon m}}{m} \left(h^{ab} e^{im(\sigma_2 - \sigma_1)} + h^{ba} e^{-im(\sigma_2 - \sigma_1)} \right), \tag{2.12}$$

where ϵ is a regulator, and

$$h^{ij} \equiv \frac{1}{g + 2\pi\alpha' B} \,. \tag{2.13}$$

As we have seen, this tensor when expanded about large B has the form:

$$h^{ij} \sim \frac{\theta^{ij}}{2\pi\alpha'} - \frac{(\theta g \theta)^{ij}}{2\pi\alpha'^2} + \cdots, \tag{2.14}$$

where the terms in the expansion are alternately antisymmetric and symmetric, and the two terms exhibited above are description-independent. In this section we neglect all but the first term above.

It follows that

$$T^{a_1\ldots a_n;\,b_1\ldots b_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n \int_0^{2\pi} \frac{d\sigma}{2\pi} \prod_{i=1}^{n} \left(h^{a_i b_i} \sum_{m=1}^{\infty} \frac{e^{-\epsilon m + i m \sigma}}{m} + h^{b_i a_i} \sum_{m=1}^{\infty} \frac{e^{-\epsilon m - i m \sigma}}{m} \right). \tag{2.15}$$

After evaluating the sum over m, the result, depending on the regulator ϵ, is

$$T^{a_1\ldots a_n;\,b_1\ldots b_n} = \frac{1}{2}\frac{1}{n!}(\alpha')^n \int_0^{2\pi} \frac{d\sigma}{2\pi} \prod_{i=1}^{n} \left(-h^{[a_i b_i]} \ln \frac{(1-e^{-\epsilon+i\sigma})}{(1-e^{-\epsilon-i\sigma})} - h^{(a_i b_i)} \ln |1-e^{-\epsilon+i\sigma}|^2 \right). \tag{2.16}$$

Here, $h^{[ab]}$ and $h^{(ab)}$ are, respectively, the antisymmetric and symmetric parts of h^{ab}.

The large-B or Seiberg-Witten limit consists of the replacements:

$$h^{[ab]} \to \frac{\theta^{ab}}{2\pi\alpha'}, \qquad h^{(ab)} \to 0\,. \tag{2.17}$$

By virtue of the fact that

$$\lim_{\epsilon\to 0} \ \ln \frac{(1-e^{-\epsilon+i\sigma})}{(1-e^{-\epsilon-i\sigma})} = i(\sigma - \pi) \tag{2.18}$$

this limit leads to an elementary integral. Evaluating it, one finally obtains [8] the result:

$$\begin{aligned} S_{CS} + \Delta S_{CS} &= \\ &= \frac{1}{2}\int C^{(6)} \wedge \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^{2j}(2j+1)!}\, \theta^{a_1 b_1} \ldots \theta^{a_{2j} b_{2j}}\, \partial_{a_1} \ldots \partial_{a_{2j}} F \wedge \partial_{b_1} \ldots \partial_{b_{2j}} F \\ &= \frac{1}{2}\int C^{(6)} \wedge \langle F \wedge F \rangle_{*_2}\,, \end{aligned} \tag{2.19}$$

where the product $*_2$ was defined in eq. (1.8). This agrees perfectly with eq. (2.4), the prediction from noncommutativity.

3. Effective action beyond the Seiberg-Witten limit

In this section, we extend the calculation from the point of eq. (2.16). To go to first order beyond the Seiberg-Witten limit, we make the replacements:

$$h^{[ab]} \to \frac{\theta^{ab}}{2\pi\alpha'}\,, \qquad h^{(ab)} \to -\frac{\theta^{ac} g_{cd} \theta^{db}}{(2\pi\alpha')^2} \tag{3.1}$$

and keep all terms that are first order in $h^{(ab)}$.

Denote by $T_{(1)}^{a_1\ldots a_n;b_1\ldots b_n}$ the first correction to T (defined in eq. (2.11)) away from the Seiberg-Witten limit. Then, we see that

$$T_{(1)}^{a_1\ldots a_n;b_1\ldots b_n} = \frac{(-i)^{n-1}}{2}\frac{1}{(n-1)!}\frac{1}{(2\pi)^n}\frac{(\theta g\theta)^{a_1b_1}}{2\pi\alpha'}\theta^{a_2b_2}\cdots\theta^{a_nb_n}\int_0^{2\pi}\frac{d\sigma}{2\pi}(\sigma-\pi)^{n-1}\ln|1-e^{i\sigma}|^2\,. \tag{3.2}$$

The integral in the above expression vanishes for even n. For odd $n = 2p+1$, we find that

$$T_{(1)}^{a_1\ldots a_{2p+1};b_1\ldots b_{2p+1}} = \frac{(-1)^p}{2}\frac{1}{(2p)!}\frac{1}{(2\pi)^{2p+1}}\frac{(\theta g\theta)^{a_1b_1}}{2\pi\alpha'}\theta^{a_2b_2}\cdots\theta^{a_{2p+1}b_{2p+1}}I_{2p+1}\,, \tag{3.3}$$

where

$$\begin{aligned} I_{2p+1} &\equiv \int_0^{2\pi}\frac{d\sigma}{2\pi}(\sigma-\pi)^{2p}\ln|1-e^{i\sigma}|^2 \\ &= 2(-1)^p(2p)!\sum_{j=0}^{p-1}(-1)^j\frac{\pi^{2j}}{(2j+1)!}\zeta(2p-2j+1)\,. \end{aligned} \tag{3.4}$$

It is convenient to define a 4-form W_4 that encodes the derivative corrections for the coupling to $C^{(6)}$:

$$S_{CS} + \Delta S_{CS} = \frac{1}{2}\int C^{(6)}\wedge F\wedge F + \int C^{(6)}\wedge W_4\,. \tag{3.5}$$

As we have seen in the previous section, the leading-order term in W_4 in the large-B limit is:

$$W_4^{(0)} = \langle\, F\wedge F\,\rangle_{*_2} - F\wedge F\,. \tag{3.6}$$

The calculations leading to eq. (3.4) amount to computing W_4 to first order (in α') around the Seiberg-Witten limit:

$$\begin{aligned} W_4^{(1)} &= \sum_{p=1}^{\infty} T_{(1)}^{a_1\ldots a_{2p+1};b_1\ldots b_{2p+1}}\partial_{a_1}\ldots\partial_{a_{2p+1}}F\wedge\partial_{b_1}\ldots\partial_{b_{2p+1}}F \\ &= \sum_{p=1}^{\infty}\frac{1}{(2\pi)^{2p+1}}\frac{(\theta g\theta)^{a_1b_1}}{2\pi\alpha'}\theta^{a_2b_2}\cdots\theta^{a_{2p+1}b_{2p+1}}\partial_{a_1}\ldots\partial_{a_{2p+1}}F\wedge\partial_{b_1}\ldots\partial_{b_{2p+1}}F\ \times \\ &\quad\times\sum_{j=0}^{p-1}(-1)^j\frac{\pi^{2j}}{(2j+1)!}\zeta(2p-2j+1)\,. \end{aligned} \tag{3.7}$$

Interchanging the order of the two summations, we find that the sum over j can be performed and leads to the appearance of the familiar $*_2$ product. The result, after some relabelling of indices, is:

$$W_4^{(1)} = \sum_{p=0}^{\infty}\frac{\zeta(2p+3)}{(2\pi)^{2p+3}}\frac{(\theta g\theta)^{cd}}{2\pi\alpha'}\theta^{a_1b_1}\cdots\theta^{a_{2p+2}b_{2p+2}}\times\langle\partial_c\partial_{a_1}\ldots\partial_{a_{2p+2}}F\wedge\partial_d\partial_{b_1}\ldots\partial_{b_{2p+2}}F\rangle_{*_2}\,. \tag{3.8}$$

Unlike the leading term eq. (2.19), which is a single infinite series in derivatives summarised by the $*_2$ product, here we see a double infinite series. After forming the $*_2$ product we still have an additional series whose coefficients are ζ-functions of odd argument.

A more elegant representation of the result can be found by replacing the ζ-functions by their series representation: $\zeta(2p+3) = \sum_{m=1}^{\infty} m^{-2p-3}$, and interchanging the m and p summations. This leads to:

$$W_4^{(1)} = \frac{(\theta g \theta)^{cd}}{2\pi\alpha'} \left\langle \partial_c F \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left(\frac{1}{1 - \left(\overleftarrow{\partial_p}\, \theta^{pq}\, \overrightarrow{\partial_q}/(2\pi m) \right)^2} - 1 \right) \wedge \partial_d F \right\rangle_{*_2} . \tag{3.9}$$

If the momenta of the two gauge fields are $k^{(1)}$ and $k^{(2)}$, then we see that the term inside brackets develops a pole whenever

$$k_p^{(1)}\, \theta^{pq}\, k_q^{(2)} = 2\pi m \tag{3.10}$$

for any positive or negative integer m. Nevertheless, $W_4^{(1)}$, and hence the effective action, remains nonsingular, because at precisely the above values of momenta, the $*_2$ product develops zeroes.

Thus we have a finite expression for the first correction to the effective action beyond the large-B limit. In the following section we will compare this with a direct computation using noncommutativity.

4. Comparison with a noncommutative amplitude calculation

The tree-level amplitude with two open-string vertex operators and one closed string vertex operator on a disk has been evaluated exactly by Liu and Michelson [18]. This calculation was performed in the $\Phi = 0$ description, which is most convenient for worldsheet computations. In this section we are going to use it to compare with results in the previous section that are of first order in α'. As we have noted, to this order the description is irrelevant.

We will now use this to extract the term corresponding to our computation in eq. (3.8) and compare the two expressions. The computation of ref. [18] is an evaluation of the amplitude:

$$\mathcal{A}_2 \equiv \int_{-\infty}^{\infty} dy \left\langle V_{RR}^{-(1/2),-(3/2)}(q;i)\, V_O^0(a_1,k_1;0)\, V_O^0(a_2,k_2;y) \right\rangle , \tag{4.1}$$

where $V_{RR}^{-1/2,-3/2}$ is the vertex operator for an RR potential of momentum q, in the $(-1/2,-3/2)$ picture, and V_O^0 are vertex operators for massless gauge fields of momentum k_i and polarizations a_i, $i=1,2$.

We define:

$$\begin{aligned} t &\equiv \alpha' k_1 \cdot k_2 = \alpha' k_{1i}\, G^{ij}\, k_{2j} \\ a &\equiv \frac{1}{2\pi} k_1 \times k_2 = \frac{1}{2\pi} k_{1i}\, \theta^{ij}\, k_{2j} \end{aligned} \tag{4.2}$$

and change integration variables via $y = -\cot\pi\tau$. Then it follows (for details, see ref. [18, section 3.2 and appendix D]) that the coefficient of $F \wedge F$ provided by this computation, to be compared with the coefficient of eq. (3.8), is:

$$2^{2t}\int_0^{1/2} d\tau\,(\cos\pi\tau)^{2t}\,\cos 2\pi a\tau = \frac{1}{2}\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)}\,. \tag{4.3}$$

The RHS can be expanded in powers of t and, up to terms of $\mathcal{O}(t^2)$, one has:

$$\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)} = \frac{1}{\Gamma(1-a)\Gamma(1+a)}\left[1-(2\gamma+\psi(1-a)+\psi(1+a))\,t+\mathcal{O}(t^2)\right], \tag{4.4}$$

where γ is the Euler constant and $\psi(x)$ is the digamma function $(d/dx)\ln\Gamma(x)$.

The first term can be recognised as the kernel of the $*_2$-product, using the relation:

$$\frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin\pi a}{\pi a}\,. \tag{4.5}$$

Let us now examine the second term more carefully. We use the fact that:

$$\psi(1+x) = -\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1} \tag{4.6}$$

to write:

$$\begin{aligned} 2\gamma+\psi(1-a)+\psi(1+a) &= -\sum_{k=2}^{\infty}\left(1-(-1)^k\right)\zeta(k)\,a^{k-1} \\ &= -2\sum_{p=0}^{\infty}\zeta(2p+3)\,a^{2p+2}\,. \end{aligned} \tag{4.7}$$

Putting everything together, we find that:

$$\begin{aligned} W_4^{(1)} &= -\frac{\sin((k_1\times k_2)/2)}{(k_1\times k_2)/2}\times \\ &\quad\times\sum_{p=0}^{\infty}\frac{\zeta(2p+3)}{2\pi\alpha'(2\pi)^{2p+3}}(k_1\times k_2)^{2p+2}\left(k_{1i}(\theta g\theta)^{ij}k_{2j}\right)\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\,. \end{aligned} \tag{4.8}$$

On Fourier transforming, this is identical to eq. (3.8) of the previous section.

5. Wilson lines and deformed $*$-products

We have seen in the previous section that the correction to the commutative Chern-Simons action found in section 3 matches with a corresponding computation in noncommutative language. On general grounds, one expects that the commutative and the noncommutative actions match precisely to all orders upon using the SW map. Therefore it is natural to ask how the correction computed in this paper can be re-expressed as an effective action in terms of the noncommutative field variables.

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As discussed in section 2, the leading term for the Chern-Simons coupling in the Seiberg-Witten limit is given in terms of straight Wilson lines by eq. (2.3). In what follows, we attempt to interpret the corrections beyond the SW limit in the language of open Wilson lines. In subsections 5.1 and 5.2, we investigate the consequences for the topological identity of refs. [4, 5, 6] and the Seiberg-Witten map beyond the SW limit. We restrict ourselves to quadratic order in gauge fields. This enables us to make use of the explicit computation in ref. [18], and also to avoid many technical complications. In order to get a complete picture one will have to go beyond the quadratic approximation, an issue which we intend to address in future work.

As we are now going to use the results of ref. [18] to all orders in the expansion around the SW limit, it will be important to work with a specific description, namely $\Phi = 0$, the one in which these calculations were performed. Henceforth it will be understood that we are always in this description.

Our first observation will be that the corrections away from the SW limit can be reproduced by modifying the prescription for the noncommutative effective action. The usual prescription, valid in the SW limit, is to smear local operators along an open Wilson line that runs along a specific straight contour. We will show that corrections can be incorporated by modifying the way in which local operators are smeared along the contour. The modification consists of inserting a specific function of the smearing parameter, and leads to a deformed $*_2$ product. Next we will find an equivalent but more suggestive way to incorporate the corrections. This uses the standard smearing prescription and Wilson line, but the Moyal $*$ product is deformed in a particular way. The relationship between deformed $*_n$ products and deformed Moyal $*$-products will be seen to arise in many contexts, which we take as an encouraging indication that this extension of the usual picture is a natural one. We will be led to conjecture that the entire large-B expansion on the noncommutative side is encoded in a deformed $*$-product or series of such products, along with the prescription of smearing over a straight Wilson line.

Let us start by reviewing how the $*_2$ product originates from smearing. The straight Wilson line runs along a contour:

$$x^i(\tau) = x^i + \theta^{ij} k_j \tau \,, \tag{5.1}$$

where $\tau \in [0, 1]$. As long as we work to quadratic order in the commutative field strength F, we can replace $\hat{F}$ by F in eq. (2.3). The only effect of the Wilson line in this case is that the operators are path-ordered and smeared along its contour. One of the inserted operators may be fixed to lie at the starting point of the contour (5.1). Thus for a product of two local operators, we only need to smear the second one linearly over the contour. In quadratic order, eq. (2.3) therefore reduces to:

$$\frac{1}{2}\widetilde{C}^{(6)}(-k) \wedge \int_0^1 d\tau \int dx\ F(x) * \wedge F(x + \theta \cdot k\tau)\, e^{ik.x} \,. \tag{5.2}$$

From now on we will drop the $\widetilde{C}^{(6)}$ factor for simplicity.

Going to momentum space,[5] we have:

$$\frac{1}{2}\int_0^1 d\tau \int dx\ dk_1\ dk_2\ e^{-(i/2)k_1\times k_2}\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\ e^{-ik_1.x}e^{-ik_2.(x+\theta\cdot k\tau)}e^{ik.x}\,. \tag{5.3}$$

The τ and x integrals are easily evaluated, leading to:

$$\frac{1}{2}\int dk_1\, dk_2\ \frac{\sin{(k_1\times k_2)/2}}{(k_1\times k_2)/2}\,\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\,\delta(k_1+k_2-k)\,. \tag{5.4}$$

This is the result that we have quoted several times in previous sections, since

$$\frac{\sin{(k_1\times k_2)/2}}{(k_1\times k_2)/2}\,\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\equiv\langle\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\rangle_{*_2}\,. \tag{5.5}$$

Eq. (5.4) agrees with the result of Liu and Michelson [18], written in eq. (4.3), and evaluated at $t=0$ using eq. (4.5). Our goal is now to reproduce all of eq. (4.3), and not just its limit at $t=0$, from a smearing prescription.

Before proceeding to do this, let us define the abstract symbol $*_2$ as shorthand for the corresponding momentum space kernel:

$$*_2\equiv\frac{\sin{(k_1\times k_2)/2}}{(k_1\times k_2)/2}=\frac{\sin\pi a}{\pi a}\,. \tag{5.6}$$

We now modify the smearing prescription by introducing a weight factor $f(t,\tau)$ into the correlation function. This amounts to replacing eq. (5.2) by:

$$\frac{1}{2}\widetilde{C}^{(6)}(-k)\wedge\int_0^1 d\tau\, f(t,\tau)\int dx\ F(x)*\wedge F(x+\theta\cdot k\tau)\,e^{ik.x}\,. \tag{5.7}$$

The function $f(t,\tau)$ will be determined by requiring agreement between the effect of this modified smearing, and the amplitude in eq. (4.3). This function should of course reduce to unity in the Seiberg-Witten limit: $f(t=0,\tau)=1$.

Going to momentum space and dropping the explicit $\widetilde{C}^{(6)}$, as before, this amounts to modifying eq. (5.4) to:

$$\frac{1}{2}\int_0^1 d\tau\, f(t,\tau)\int dk_1\ dk_2\ e^{-(i/2)k_1\times k_2}\widetilde{F}(k_1)\wedge\widetilde{F}(k_2)\ e^{i(k_1\times k_2)\tau}\,. \tag{5.8}$$

The coefficient of $F\wedge F$ in this expression will become equivalent to eq. (4.3) if we make the choice:

$$f(t,\tau)=(2\sin\pi\tau)^{2t}\,. \tag{5.9}$$

This follows by writing the τ-integral in eq. (5.8) as:

$$\frac{1}{2}\int_0^1 d\tau\,(2\sin\pi\tau)^{2t}\,e^{-i\pi a}\ e^{2\pi i a\tau}=\int_0^{1/2} d\tau\,(2\cos\pi\tau)^{2t}\ \cos 2\pi a\tau \tag{5.10}$$

which is precisely the LHS of eq. (4.3).

[5] The integration measures dx, dk_i are defined to implicitly include the necessary factors of 2π.

It follows that, to quadratic order in F, the complete tree-level derivative correction to the CS coupling (5.2) around the SW limit is given by the insertion of $(2\sin\pi\tau)^{2t}$ in the smearing prescription.

We would now like to cast this in an equivalent form which will be more useful. The idea is to find a t-dependent deformation of the Moyal $*$-product in such a way that the full coupling is again given by (5.2) but with the new $*$-product replacing the old one. For this, let us first recall that the kernel which gave rise to the $*_2$ product, on the LHS of eq. (4.4) is:

$$\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)} \sim \frac{\sin\pi a}{\pi a} + \mathcal{O}(t)\,. \tag{5.11}$$

Now instead of expanding in t, we rewrite the exact result as:

$$\frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)} = \frac{\sin\pi a}{\pi a}\left(\frac{\Gamma(1+a)\Gamma(1-a)\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)}\right). \tag{5.12}$$

Since the LHS of the above equation reduces to the $*_2$ product when $t=0$, it is tempting to think of it, for general t, as a deformation of $*_2$. Thus, generalizing eq. (5.6), we define:

$$*_2\,(t) \equiv \frac{\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)}\,. \tag{5.13}$$

Notice that $*_2(t)$ is also symmetric under $a\to -a$ (that is under $k_1 \leftrightarrow k_2$). It follows that:

$$*_2\,(t) = *_2 \times \mathcal{K}(t)\,, \tag{5.14}$$

where

$$\mathcal{K}(t) \equiv \frac{\Gamma(1+a)\Gamma(1-a)\Gamma(1+2t)}{\Gamma(1+a+t)\Gamma(1-a+t)}\,. \tag{5.15}$$

In terms of the deformed $*_2$ product, we can write the exact tree-level coupling of $C^{(6)}$ to two F's, as derived in ref. [18], as:

$$\frac{1}{2}\widetilde{C}^{(6)}(-k)\wedge\int dx\,\langle F\wedge F\rangle_{*_2(t)}\,e^{ik.x}\,. \tag{5.16}$$

By construction, the new kernel $*_2(t)$ has a power series expansion in t, of which the zeroth order term is the usual $*_2$ kernel. Thus we can define a sequence of kernels via the power series:

$$\begin{aligned} *_2(t) &= \sum_{n=0}^{\infty} *_2^{(n)}\;t^n = *_2 + *_2^{(1)}\;t + \mathcal{O}(t^2) \\ \mathcal{K}(t) &= \sum_{n=0}^{\infty} \mathcal{K}^{(n)}\;t^n = 1 + \mathcal{K}^{(1)}\;t + \mathcal{O}(t^2)\,, \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} *_2(t=0) &= *_2^{(0)} = *_2 \\ \mathcal{K}(t=0) &= \mathcal{K}^{(0)} = 1\,. \end{aligned} \tag{5.18}$$

The next step is to define a generalized t-dependent Moyal product. Recall that the Moyal $*$-product and $*_2$ kernel are related via the following equation:

$$\int dx \int_0^1 d\tau\, (f(x) * g(x + \theta \cdot k\tau))\; e^{ik.x} = \int dx\; \langle f, g\rangle_{*_2}\, e^{ik.x}\,. \tag{5.19}$$

This can be generalized in a way that produces the full $*_2(t)$ on the RHS. Namely, we require a generalization of the Moyal $*$, called $*(t)$, that satisfies:

$$\int dx \int_0^1 d\tau\, \Big(f(x) *(t)\; g(x + \theta \cdot k\tau)\Big)\; e^{ik.x} = \int dx\; \langle f, g\rangle_{*_2(t)}\, e^{ik.x}\,. \tag{5.20}$$

It is easy to check that the solution for $*(t)$ is given by:

$$*(t) \;\equiv\; * \times \mathcal{K}(t) \tag{5.21}$$

with $\mathcal{K}(t)$ given in eq. (5.15).

With these definitions, one can write the modified smearing prescription of eq. (5.7) as an ordinary smearing over the Wilson line (with no extra function $f(t,\tau)$), but using the generalized Moyal product $*(t)$:

$$\frac{1}{2}\widetilde{C}^{(6)}(-k) \wedge \int_0^1 d\tau \int dx\; F(x) *(t) \wedge F(x + \theta \cdot k\tau)\, e^{ik.x}\,. \tag{5.22}$$

By virtue of eq. (5.20), this is identical to the desired result in eq. (5.16) above. Thus we have been led first to a t-dependent $*_2(t)$ product and thence to a Moyal product $*(t)$, in terms of which the Chern-Simons coupling of two gauge fields to the RR 6-form can be succinctly expressed. This expression is exact at open string tree level.

This observation can be followed up in two different directions. On the one hand, the modified products above will affect the key results of ref. [4, 5, 6], namely the topological identity and the Seiberg-Witten map obtained from a comparison of commutative and noncommutative CS couplings. On the other hand, it is important to check whether the structure of deformed $*$ products remains relevant when one goes beyond quadratic order in the gauge field. We now turn to the first of these issues, reserving the second for future work.

5.1 Topological identity revisited

In this subsection we reconsider the topological identity of refs. [4, 5, 6]. This was originally discovered by comparing the noncommutative expression for the 10-form coupling of a D9-brane in the SW limit with the corresponding commutative one.

We know that in the SW limit $t \to 0$, the noncommutative 10-form coupling to a D9-brane is:

$$\widetilde{C}^{(10)}(-k) \int dx\; L_* \left[\sqrt{\det(1 - \theta\hat{F})}\; W(x, C)\right] * e^{ik.x}\,. \tag{5.23}$$

On the other hand, the commutative coupling is:

$$\widetilde{C}^{(10)}(-k) \int dx\; e^{ik.x}\,. \tag{5.24}$$

This receives no derivative corrections, as shown in ref. [12]. Equating the two expressions above leads to the topological identity:

$$\int dx\ L_* \left[\sqrt{\det(1-\theta\hat{F})}\ W(x,C)\right] * e^{ik.x} = \delta(k)\,. \tag{5.25}$$

If we go beyond the SW limit, the commutative 10-form coupling still receives no corrections, since the argument of ref. [12] does not depend on this limit at all. On the other hand, the noncommutative coupling would be changed according to our prescription above for t-deformed $*$-products. Therefore we need to check that the equivalence of the two couplings continues to hold beyond the SW limit, as it certainly should.

Let us first review some aspects of the above identity. For it to hold, all terms in the noncommutative action must cancel out, to every order in $\hat{A}$, except for the zeroth order term. Explicit computations upto cubic order in $\hat{A}$ were presented in ref. [4].[6] The cancellation to first order in $\hat{A}$ is very straightforward, since no smearing is required. Among the quadratic terms, we have a contribution:

$$\frac{i}{2}\,\theta^{ij}[\hat{A}_j,\hat{A}_i]_* \tag{5.26}$$

coming from the term of first order in $\hat{F}$ from the pfaffian, and using the definition:

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i[\hat{A}_i,\hat{A}_j]_*\,. \tag{5.27}$$

This term has no smearing, since the smearing prescription requires us to treat a commutator of gauge fields as a single object.

Using the identity:

$$[f,g]_* = i\,\theta^{kl}\langle\partial_k f,\partial_l g\rangle_{*_2} \tag{5.28}$$

this term can also be written:

$$\frac{1}{2}\,\theta^{ij}\theta^{kl}\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2}\,. \tag{5.29}$$

There is also a term, coming from quadratic order in the expansion of the pfaffian, which is:

$$-\frac{1}{2}\,\theta^{ij}\theta^{kl}\,\partial_k\hat{A}_i(x) * \partial_l\hat{A}_j(x+\theta\cdot k\tau)\,. \tag{5.30}$$

This term needs to be integrated over τ. As one can check, all the other terms to quadratic order in $\hat{A}$ cancel each other out identically *even before* τ-integration. Thus, combining eqs. (5.29) and (5.30), we are left with a quadratic contribution:

$$\frac{1}{2}\,\theta^{ij}\theta^{kl}\int_0^1 d\tau\ \Big(\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2} - \partial_k\hat{A}_i(x) * \partial_l\hat{A}_j(x+\theta\cdot k\tau)\Big) \tag{5.31}$$

to the noncommutative 10-form coupling. The first term in this bracket is independent of τ, but it can be put inside the τ integral anyway.[7] In this form, it is clear that upon

[6] A proof of the topological identity in the SW limit using the matrix model language can be found in [5].
[7] Also, here and in what follows we are suppressing the factors $e^{ik.x}$ and $\int dx$.

integrating over τ, the second term gets converted to $*_2$ and cancels the first. Hence the quadratic terms cancel out as expected.

The question of interest now is what happens to this cancellation when we go beyond the Seiberg-Witten limit by modifying the $*$ product. First note that for all the other quadratic terms, which we have been ignoring because they cancelled pointwise in τ, the cancellation continues to hold even if we replace the Moyal $*$ product by $*(t)$. Thus we only need to concentrate on the terms in eq. (5.31) above, which generalize to:

$$\frac{1}{2}\,\theta^{ij}\theta^{kl}\int_0^1 d\tau\ \Big(\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2} - \partial_k\hat{A}_i(x)*(t)\ \partial_l\hat{A}_j(x+\theta\cdot k\tau)\Big)\,. \tag{5.32}$$

Using eq. (5.20), this can be rewritten:

$$\frac{1}{2}\,\theta^{ij}\theta^{kl}\ \Big(\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2} - \langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2(t)}\Big) \tag{5.33}$$

and clearly this is no longer zero.

Now it is clear what must be done in order to restore the cancellation of quadratic terms. We somehow need to modify the first term above by the replacement:

$$\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2} \to \langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2(t)}\,, \tag{5.34}$$

where $*_2(t)$ is the t-deformed $*_2$ kernel defined in eq. (5.13). If this can be done, then the equivalence of commutative and noncommutative 10-form couplings will be restored. Otherwise the approach we are proposing will fail to work for the 10-form coupling, and we will simply not be able to find a general prescription to go beyond the Seiberg-Witten limit along these lines.

Recall where this first term originated — it came from the Moyal commutator in the gauge field strength $\hat{F}$. So the above replacement can be achieved only if we change the definition of the noncommutative field strength $\hat{F}$, to:

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i + \theta^{kl}\langle\partial_k\hat{A}_i\,\partial_l\hat{A}_j\rangle_{*_2(t)}\,. \tag{5.35}$$

In order for the last term to be a commutator of some generalized product, we need an identity analogous to eq. (5.28). Happily, the required identity

$$[f,g]_{*(t)} = i\,\theta^{kl}\langle\partial_k f,\partial_l g\rangle_{*_2(t)} \tag{5.36}$$

can be shown to hold, as a consequence of eq. (5.20) which was our defining equation for the t-deformed Moyal product. It follows that the correct redefinition of the field strength is:

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i[\hat{A}_i,\hat{A}_j]_{*(t)}\,. \tag{5.37}$$

This amounts to a consistency check. We have learned that upon replacing the Moyal $*$-product by the expression $*(t)$ *both* in the action *and* in the definition of $\hat{F}$, the equivalence of 10-form CS couplings is restored. Another way of saying this is that the two (in principle independent) ways of defining $*(t)$, via eqs. (5.20) and (5.36) are equivalent. Thus our proposal has passed a crucial test.

This immediately brings up the question of whether our proposal to deform the $*$ product is in conflict with the theorem of Kontsevich [22], which essentially states that the unique associative, noncommutative product up to certain "gauge freedoms", is the Moyal $*$-product. The answer is that we do not claim $*(t)$ to be associative. In fact, it is easily checked that $*(t)$ is non-associative[8] except in the limit $t \to 0$. We have defined the deformed Moyal $*$-product only for a pair of functions, since we are working to quadratic order in $\hat{A}$, so it is not even clear that one should ask questions about associativity at this stage. When considering higher order amplitudes, this will clearly be an important issue to understand.

5.2 Seiberg-Witten map revisited

We now address the question of how our deformations of $*_2$ and the Moyal $*$ product affect the Seiberg-Witten map. This map is the defining equation for a commutative gauge field in terms of a noncommutative one (or vice-versa). One way of obtaining the SW map is by comparing the expression for the coupling of a noncommutative Dp-brane to the $C^{(p-1)}$ form with its commutative counterpart [4, 5, 6]. This method relies crucially on the fact [12] that one can always choose a basis of fields on the commutative side, in terms of which the coupling $\int C^{(p-1)} \wedge F$ does not receive any derivative corrections. Writing the noncommutative coupling in the SW limit and equating it to the commutative coupling, one gets:

$$\widetilde{F}_{ij}(k) = \int dx \; L_* \Big[\sqrt{\det(1-\theta\hat{F})}\, (\hat{F}(1-\theta\hat{F})^{-1})_{ij}\, W(x,C) \Big] * e^{ik.x}\,, \tag{5.38}$$

where $\widetilde{F}_{ij}(k)$ is the Fourier transform of F_{ij}. The commutative field strength F_{ij} satisfies the Bianchi Identity

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0\,. \tag{5.39}$$

The RHS of (5.38) is, by construction, invariant under the noncommutative gauge transformations whose infinitesimal form is given by:

$$\hat{A}_i \to \hat{A}_i + \partial_i\hat{\lambda} - i[\hat{A}_i, \hat{\lambda}]_*\,. \tag{5.40}$$

Based on our experience with the topological identity in the previous section, we now propose a modification of the SW map upto terms of $\mathcal{O}(\hat{A}^3)$ which will be valid perturbatively to all orders in the SW expansion.

Recall that, to lowest order in the SW expansion and to $\mathcal{O}(\hat{A}^2)$, we have:

$$F_{ij} = \hat{F}_{ij} + \theta^{mn}\langle \hat{F}_{im}, \hat{F}_{nj}\rangle_{*_2} - \frac{1}{2}\theta^{mn}\langle \hat{F}_{nm}, \hat{F}_{ij}\rangle_{*_2} + \theta^{mn}\partial_n\langle \hat{A}_m, \hat{F}_{ij}\rangle_{*_2} + \mathcal{O}(\hat{F}^3)\,. \tag{5.41}$$

Consistent with our earlier results, it is natural to propose that this expression is modified by the replacement of $*_2$ products by $*_2(t)$ products and by redefining $\hat{F}_{ij}$ to be:

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i + \theta^{kl}\langle \partial_k\hat{A}_i, \partial_l\hat{A}_j\rangle_{*_2(t)}\,. \tag{5.42}$$

[8]Non-associative products generalizing the Moyal $*$ were considered, in a different context, in ref. [23]. See also [24].

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As we have seen in the case of topological identity, both these changes can be achieved by the replacement $* \to *(t)$ inside the open Wilson line and in the definition of $\hat{F}$. With these two changes, eq. (5.41) is modified to:

$$F_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \theta^{mn}\Big[\langle \partial_i \hat{A}_m, \partial_n \hat{A}_j\rangle_{*_2(t)} + \langle \partial_m \hat{A}_i, \partial_j \hat{A}_n\rangle_{*_2(t)} - \langle \partial_i \hat{A}_m, \partial_j \hat{A}_n\rangle_{*_2(t)}\Big] + \\ + \theta^{mn}\langle \hat{A}_m, \partial_n(\partial_i \hat{A}_j - \partial_j \hat{A}_i)\rangle_{*_2(t)} + \mathcal{O}(\hat{A}^3)\,. \tag{5.43}$$

Notice that this expression for $F_{ij}(\hat{A}_i)$ has exactly the same form as its $t = 0$ counterpart except for the change of $*_2 \to *_2(t)$. We claim that this is the correct generalization of the SW map (5.41) to $\mathcal{O}(\hat{A}^2)$ and perturbatively to all orders in the SW expansion.

This can be checked in various ways. For example, it can be shown that this is precisely the expression for the SW map that one obtains by comparing the commutative and noncommutative couplings of a Dp-brane to the RR $C^{(p-1)}$ form in the presence of a constant B-field. We present this argument, which uses the calculations of ref. [18], in the appendix.

Another check is to show explicitly that the above modification of the SW map does satisfy the Bianchi identity eq. (5.39). This turns out to be quite straightforward, so we omit the calculation here. On the other hand, eq. (5.43) is no longer invariant under the gauge transformations in eq. (5.40). To achieve gauge invariance, we find, perhaps not surprisingly, that we have to also promote the $*$-product in eq. (5.40) to a $*(t)$-product. Thus the new gauge transformation law for the noncommutative gauge field $\hat{A}_i$ is:

$$\hat{A}_i \to \hat{A}_i + \partial_i \hat{\lambda} + \theta^{kl}\langle \partial_k \hat{A}_i, \partial_l \hat{\lambda}\rangle_{*_2(t)}\,. \tag{5.44}$$

Though the validity of the Bianchi identity proves that the LHS of (5.43) is a $U(1)$ gauge field, it is nevertheless instructive to actually construct explicitly the gauge transformation of the commutative gauge potential out of the noncommutative gauge transformation law eq. (5.44). Let us first review what is known about this in the $t = 0$ case. A solution to the differential equations of ref. [9] for the SW map to quadratic order can be found in the paper of Mehen and Wise (the third reference in [11]). Their solution, upto $\mathcal{O}(\hat{A}^3)$ terms, is:

$$A_i = \hat{A}_i + \frac{1}{2}\theta^{mn}\langle \hat{A}_m, (2\partial_n \hat{A}_i - \partial_i \hat{A}_n)\rangle_{*_2} + \mathcal{O}(\hat{A}^3)\,. \tag{5.45}$$

One can check that this is a correct solution by evaluating $F_{ij} = \partial_i A_j - \partial_j A_i$ and comparing with what one gets from the open Wilson line prescription (essentially eq. (5.43) with $*_2(t) \to *_2$). The way Seiberg and Witten set up their equations ensures, by construction, that the variation of RHS of (5.45) under:

$$\hat{A}_i \to \hat{A}_i + \partial_i \hat{\lambda} + \theta^{kl}\langle \partial_k \hat{A}_i, \partial_l \hat{\lambda}\rangle_{*_2} \tag{5.46}$$

amounts to the commutative gauge transformation $A_i \to A_i + \partial_i \lambda$, provided the two parameters λ and $\hat{\lambda}$ are related to each other by:

$$\lambda = \hat{\lambda} + \frac{1}{2}\theta^{mn}\langle \hat{A}_m, \partial_n \hat{\lambda}\rangle_{*_2} + \mathcal{O}(\hat{A}^2\hat{\lambda})\,. \tag{5.47}$$

One can easily check that $\delta_{\hat{\lambda}} A_i(\hat{A}) = \partial_i \lambda$ using (5.46), (5.45) and (5.47). For the sake of completeness, the variation $\delta_{\hat{\lambda}} A_i(\hat{A})$ of eq. (5.45) is:

$$\begin{aligned}
\delta_{\hat{\lambda}} A_i &= \partial_i \hat{\lambda} + \theta^{mn} \langle \partial_m \hat{A}_i, \partial_n \hat{\lambda} \rangle_{*_2} + \frac{1}{2}[\langle \hat{A}_m, \partial_n \partial_i \hat{\lambda} \rangle_{*_2} + \langle \partial_m \hat{\lambda}, (2\partial_n \hat{A}_i - \partial_i \hat{A}_n) \rangle_{*_2}] + \mathcal{O}(\hat{A}^2 \hat{\lambda}) \\
&= \partial_i [\hat{\lambda} + \frac{1}{2} \theta^{mn} \langle \hat{A}_m, \partial_n \hat{\lambda} \rangle_{*_2} + \mathcal{O}(\hat{A}^2 \hat{\lambda})] = \partial_i \lambda \,. \qquad (5.48)
\end{aligned}$$

To go beyond the SW limit, we again replace all $*_2$-products in the above calculation by $*_2(t)$-products. This actually achieves the required generalization. Checking this is again trivial, as the calculation is essentially identical to the one in the $t = 0$ case, with every $*_2$ replaced by $*_2(t)$.

To summarize, the $t \neq 0$ SW map at the level of the gauge potential, upto $\mathcal{O}(\hat{A}^3)$ terms, is:

$$A_i = \hat{A}_i + \frac{1}{2} \theta^{mn} \langle \hat{A}_m, (2\partial_n \hat{A}_i - \partial_i \hat{A}_n) \rangle_{*_2(t)} + \mathcal{O}(\hat{A}^3) \,. \qquad (5.49)$$

The gauge transformation of the noncommutative gauge field is:

$$\hat{A}_i \to \hat{A}_i + \partial_i \hat{\lambda} + \theta^{kl} \langle \partial_k \hat{A}_i, \partial_l \hat{\lambda} \rangle_{*_2(t)} \,. \qquad (5.50)$$

This generates the commutative gauge transformation $A_i \to A_i + \partial_i \lambda$ provided the two parameters λ and $\hat{\lambda}$ are related by:

$$\lambda = \hat{\lambda} + \frac{1}{2} \theta^{mn} \langle \hat{A}_m, \partial_n \hat{\lambda} \rangle_{*_2(t)} + \mathcal{O}(\hat{A}^2 \hat{\lambda}) \,. \qquad (5.51)$$

6. Conclusions

In this paper we have first computed derivative corrections, in the presence of a constant B-field, to the commutative coupling of a D9-brane to the RR $C^{(6)}$ form, using boundary-state techniques. This correction term is of first order beyond the SW limit, but restricted to quadratic order in gauge fields. Next we compared it with the expansion of a corresponding world-sheet calculation of the noncommutative coupling in ref. [18], and found agreement.

Subsequently, we argued that the corrections considered here still admit an interpretation in terms of straight open Wilson lines on the noncommutative side. This provides us with a generalization of the $*_2$ product to all orders in the SW expansion, which we call $*_2(t)$, where t is essentially the scalar product of the momenta of the two gauge fields involved. There are two equivalent ways of defining the $*_2(t)$ product from the point of view of an open Wilson line. One is to smear the two operators along the length of the Wilson line using a modified smearing prescription (by inserting a nontrivial t-dependent smearing function). This definition is natural from the point of view of the open-string world-sheet. The other way is to use the same smearing prescription as in $t = 0$, but to use a deformed Moyal $*$-product, which we denote by $*(t)$. By looking at the topological identity, we learned that the definition of the noncommutative gauge field strength also needs to be changed correspondingly. We explored the consequences of our observations to the SW map, again to quadratic order in the gauge fields.

This raises a number of interesting questions. Perhaps the most important is how to generalize our results beyond quadratic order in the gauge fields. This involves correctly defining the generalizations of the higher $*_n$ products. A natural hint comes from the world-sheet correlators. This suggests that one needs to insert a factor of $|\sin \pi\tau_{ab}|^{2t_{ab}}$ with $t_{ab} = \alpha' k_a \cdot k_b$ and $\tau_{ab} = \tau_a - \tau_b$, for each pair of operators on the Wilson line at the positions τ_a and τ_b. We do not know if this can then be re-expressed in terms of the $*(t)$ products as well. It will be interesting to understand the $*(t)$-product better and find a natural place for it in string theory.

It is known that the noncommutative actions in the SW limit arise naturally from matrix theory considerations [20, 21, 16]. For example the $*_n$ products of [2] arise from the symmetrised trace prescription of the matrix model action. It is natural to ask what the deformed $*_2(t)$ product means from the matrix theory language. Since matrix theory naturally leads to the $\Phi = -B$ description [16], this will require understanding how to extend our results in section 5 to general descriptions. This is also an interesting question in its own right.

Finally, once we go beyond quadratic order in noncommutative gauge fields, the transcription of these results into commutative gauge field variables would teach us more about the structure of derivative corrections to commutative D-brane actions, generalizing the results in ref. [7]. We hope to address some of these questions in future work.

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A. World-sheet derivation of (5.41)

In sections 5.1 and 5.2, to satisfy the topological identity and to have a sensible definition of the SW map we proposed to replace the $*_2$ products by $*_2(t)$ products and also modify the definition of the noncommutative field strength $\hat{F}$ as in eq. (5.42). In this appendix, we would like to justify these proposals by looking at the the 1+1 and 1+2–point world-sheet correlation functions on the disc. Fortunately for us, these amplitudes have been evaluated in ref. [18]. The details of these calculations can be found in ref. [18, section 3] and will not be reproduced here.

We first note that the correlation functions of one closed-string vertex operator and two open-string vertex operators (as in (4.1)) involve essentially three types of integrals, [18, eqs. (3.15), (3.16) and (3.17)]. All three integrals can be evaluated explicitly and are given in of [18, eqs. (D.1a), (D.1b) and (D.1c)]. The proposal of replacing $*_2$ by $*_2(t)$ is recovered by noticing that all these integrals are proportional to the $*_2(t)$ kernel given in eq. (5.13), up to some simple factors depending on k_1 and k_2. Therefore the factor

$(\sin\frac{k_1\times k_2}{2})/(\frac{k_1\times k_2}{2})$ in of [18, eq. (3.16)] is actually the $t=0$ part of a full $*_2(t)$ kernel. This proves our first proposal.

Now let us analyze carefully how to recover (5.42) from these amplitudes. For this it is helpful to first trace back to where the commutator term in $\hat{F}$ at $t=0$ comes from, in ref. [18]. We find ourselves at the last term of eq. (3.16) of that paper, and the subsequent discussion where the authors identify this as the term that combines with the 1+1–point function contribution to give rise to a noncommutative field strength f_{MN}. Now, this term actually came from the last term of (3.12), the expression for I_{42}. This is:

$$I_{42}=\cdots+2\alpha'^2\,(k_1\cdot k_2)\frac{A_2(y)}{y(1+y^2)}\,a_{1M}a_{2N}\Lambda^{MN}\,. \tag{A.1}$$

In ref. [18] the amplitude was evaluated only to leading order in t, using (3.15b). But one could use the full answer for this integral given in (eq. (D.1b)) of the appendix, which reads:

$$\int_{-\infty}^{\infty}dy\,\frac{A_2(y)}{y(1+y^2)}=-\frac{(k_1\times k_2)}{2\alpha' k_1\cdot k_2}\times *_2(t)\,. \tag{A.2}$$

Substituting this back into of ref. [18, eq. (3.7)] it is easy to see that this converts the last term of their eq. (3.16) into the one we require. That is, the factor $(\sin k_1\times k_2/2)$ gets replaced by $((k_1\times k_2/2)\times *_2(t))$.

One more point to note is the following. There are two different sets of terms of the kind eq. (A.2) from open Wilson lines. One is the set of commutator terms from field strengths, which are smeared along the contour of the Wilson line as single objects. The other set of terms comes from the Wilson line itself, when expanded to quadratic order in the gauge fields. Now, in the way the terms in the amplitudes of ref. [18] are written, all the powers of a_N (the polarization vector of the noncommutative gauge field) that come from expanding the open Wilson line come with appropriate factors of $\mathcal{M}_i$. The one we just looked at is not one of those, and therefore is genuinely a term that comes from expanding the pfaffian and other factors in the open Wilson line which involve the field strengths rather than potentials. Therefore we conclude that when α' corrections are retained, the definition of f_{MN} in [18, eq. (3.17)] changes to the one in our eq. (5.42).

Since we have not specified what RR form we are working with (the Λ 's have a formal sum over all RR forms), our conclusion holds for all such couplings and in particular the couplings to $C^{(p+1)}$ and $C^{(p-1)}$, which are relevant for the topological identity and the SW map respectively. This completes the world-sheet proofs of our generalizations of the topological identity and redefinition of the SW map to quadratic order.

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